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The group structure of non-Abelian NS-NS transformations

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ABSTRACT

We study the transformations of the worldvolume fields of a system of multiple coinciding D-branes under gauge transformations of the supergravity Kalb-Ramond field. We find that the pure gauge part of these NS-NS transformations can be written as a $U(N)$ symmetry of the underlying Yang-Mills group, but that in general the full NS-NS variations get mixed up non-trivially with the $U(N)$. We compute the commutation relations and the Jacobi identities of the bigger group formed by the NS-NS and $U(N)$ transformations.

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1 Introduction

Although the new physics associated with systems of multiple coinciding D-branes has been known for more than a decade and a half [1], there are still a few open issues concerning the effective action that describes the low-energy dynamics of the system. One of the famous unsolved problems is the construction of a non-Abelian Born-Infeld action (see for example [2] and references thereof). The form of the Chern-Simons action is much better known, through the work of [3]-[11]. Still, also here are a few open issues left, such as the gauge invariance of the action. The invariance under $U(N)$ gauge symmetry is straightforward to show, as the entire action is written in terms of objects that transform in the adjoint representation and is traced over all Yang-Mills indices. However, the Chern-Simons term is also the one responsible for the couplings of the multiple D-brane system to the background fields of supergravity and the invariance of this part of the action under the gauge transformations of the supergravity background field has drawn little or no attention.

A few early attempts to prove the gauge invariance were made in [4, 12], but the first systematic approach was done is a series of papers by J. Adam et al., proving the explicit invariance under Ramond-Ramond (R-R) and massive gauge transformations [13, 14], under Neveu-Schwarz-Neveu-Schwarz (NS-NS) transformations [15] and unifying it in a global picture in [16].

One of the remarkable results of these papers is that the NS-NS gauge transformations are much more complicated than the R-R or the massive gauge transformations, as they not only affect the supersymmetry background fields, but also the worldvolume fields living on the D-branes. The reason why this is so, is a mixture of Abelian and non-Abelian effects: already in the case of a single (Abelian) D-brane, the Born-Infeld vector transforms with the pullback of the gauge parameter Σ_μ of the NS-NS transformation of the Kalb-Ramond field $B_{\mu\nu}$. However in the non-Abelian case this pullback is performed with $U(N)$ covariant derivatives, which under T-duality generates non-trivial transformation rules for the embedding scalars, in very much the same way as the dielectric couplings to higher-form R-R potentials appear in the action of multiple coinciding D-branes [11].

This leads immediately to a second issue: where the R-R and massive transformations are merely a straightforward generalization of their Abelian counterparts in the single D-brane system, the non-Abelian NS-NS transformation rules involve non-trivial commutator terms that are not present in the Abelian case [14, 15].

Even though the explicit invariance of the non-Abelian Chern-Simons term of the effective worldvolume action under the NS-NS transformations is proven in [15], the complicated transformation rules remain surprising and the aim of this letter is to study their structure in more detail. The main result of this paper is that the non-Abelian NS-NS transformations no longer have a $U(1)$ group structure, but that they intertwine non-trivially with the $U(N)$ Yang-Mills group of the worldvolume theory. We will show that part of the NS-NS transformations can be written as a $U(N)$ gauge transformation, but that there is also a non-trivial part that can not. We will construct explicitly the algebra spanned by the NS-NS and the $U(N)$ transformations by computing the commutation relations and the Jacobi identities.

The organization of this paper is as follows: in section 2 we will derive the NS-NS transformation rules for the worldvolume fields, following the argument of [15]. In section 3 we will construct and comment the NS-NS rules for matrix functions and composite objects such as commutators and covariant derivative, needed in the next sections. In section 4 we will construct the algebra formed by the NS-NS and $U(N)$ transformations and check the Jacobi

identities in section 5. We also review some useful issues of non-commutative algebra in the appendix A. Finally, we summarize our results in the conclusions.

2 The non-Abelian NS-NS transformation rules

In this section we will review quickly the derivation of the NS-NS transformation rules for the worldvolume fields, as done in [15], based on the T-duality between Dp- and D$(p-1)$-brane actions. The field content of a set of N multiple coinciding Dq-branes consists of a $(q+1)$-dimensional Yang-Mills (Born-Infeld) vector V_a, that acts as the gauge field of the $U(N)$ symmetry group of the the system, and a set of $9-q$ matrix-valued transverse scalars X^i, that transform in the adjoint representation of $U(N)$. From the target space point of view, the latter have the interpretation of the non-Abelian embedding scalars of the Dp-branes, while the former arises as the potential caused by the charged endpoints of open strings ending on the branes.

The T-duality between the Dp-brane and the D$(p-1)$-brane system states that the physical content of both theories is equivalent. This equivalence can be seen through an explicit mapping of the degrees of freedom of the two theories, that transforms one action in the other. We will denote the $9-p$ embedding scalars and the $(p+1)$-dimensional Born-Infeld vector of the Dp-brane by Y^i and $\hat{V}_{\hat{a}}$, and the $10-p$ embedding scalars and the p-dimensional Born-Infeld vector of the D$(p-1)$-brane by $X^{\hat{\imath}}$ and V_a respectively. Then the T-duality rules that relate the field contents of both theories after dualising in a worldvolume direction x of the Dp-brane are given by [17]

$$\begin{aligned} \hat{V}_a &\longrightarrow V_a, & Y^i &\longrightarrow X^i, \\ \hat{V}_x &\longrightarrow X^{\underline{x}}, & Y^{\underline{x}} &= \sigma^x, \end{aligned} \tag{2.1}$$

where we have decomposed $\hat{V}_{\hat{a}}$ and $X^{\hat{\imath}}$ as $\hat{V}_{\hat{a}} = (\hat{V}_a, \hat{V}_x)$ and $X^{\hat{\imath}} = (X^i, X^{\underline{x}})$ and σ^x is the worldvolume coordinate of the Dp-brane in the x-direction. The last equation is then merely an expression of the fact that we write the actions in the static gauge, at least the direction in which the T-duality is performed, while the first two state that the BI vector components and the transverse scalars in directions different from the T-dualised one are the same in both actions. The non-trivial part of the T-duality rules is contained in the third equation, that matches the degrees of freedom of the two theories by mapping the x-component of the Dp-brane BI vector with the extra embedding scalar of the D$(p-1)$-brane [17].

Of course, consistency requires that not only the degrees of freedom of both actions match according to (2.1), but also their variations. In the Abelian limit this can easily be shown to be the case. We know that on the one hand the (Abelian) Born-Infeld vector $\hat{V}_{\hat{a}}$ transforms as a vector under general coordinate transformations $\hat{\zeta}^{\hat{a}}$ in the worldvolume (i.e. reparametrisations of the worldvolume directions), as a gauge field under $U(1)$ transformations $\hat{\chi}$ and with the pull-back of a shift under NS-NS gauge transformations of the supergravity Kalb-Ramond field $\delta B_{\hat{\mu}\hat{\nu}} = 2\partial_{[\hat{\mu}}\Sigma_{\hat{\nu}]}$:

$$\delta \hat{V}_{\hat{a}} \;=\; \hat{\zeta}^{\hat{b}}\,\partial_{\hat{b}}\hat{V}_{\hat{a}} \;+\; \partial_{\hat{a}}\hat{\zeta}^{\hat{b}}\,\hat{V}_{\hat{b}} \;-\; \partial_{\hat{a}}\hat{\chi} \;-\; \Sigma_{\hat{\mu}}\,\partial_{\hat{a}}Y^{\hat{\mu}}. \tag{2.2}$$

On the other hand, the Abelian embedding scalars transform as scalars under worldvolume coordinate transformations and as target space coordinates under target space diffeomorphisms $\delta x^{\hat{\mu}} = -\xi^{\hat{\mu}}$:

$$\delta X^{\hat{\mu}} \;=\; \zeta^b\,\partial_b X^{\hat{\mu}} \;-\; \xi^{\hat{\mu}}. \tag{2.3}$$

It can easily be checked that indeed not only the field content transforms as in (2.1), but also their variations dualise as

$$\delta\hat{V}_a \longrightarrow \delta V_a, \qquad \delta\hat{V}_x \longrightarrow \delta X^{\underline{x}}, \qquad \delta Y^i \longrightarrow \delta X^i, \tag{2.4}$$

provided that the transformation parameters map to each other as ($\hat{\mu} = (\mu, \underline{x})$)

$$\begin{aligned} &\hat{\zeta}^a \longrightarrow \zeta^a, && \Sigma_\mu \longrightarrow \Sigma_\mu, && \hat{\chi} \longrightarrow \chi \ + \ \Sigma_{\underline{x}} X^{\underline{x}}, \\ &\hat{\zeta}^x \longrightarrow \Sigma_{\underline{x}}, && \Sigma_{\underline{x}} \longrightarrow \xi^{\underline{x}}. && \end{aligned} \tag{2.5}$$

The non-Abelian case is a bit more involved: here the non-Abelian Born-Infeld vector still transforms as a vector under worldvolume reparametrisations, but is now promoted to a $U(N)$ Yang-Mills gauge field and, more importantly, the pullback of the NS-NS parameter $\Sigma_{\hat{\mu}}$ needs now to be done through $U(N)$-covariant derivatives:

$$\delta\hat{V}_{\hat{a}} \ = \ \hat{\zeta}^{\hat{b}}\, \partial_{\hat{b}} \hat{V}_{\hat{a}} \ + \ \partial_{\hat{a}} \hat{\zeta}^{\hat{b}}\, \hat{V}_{\hat{b}} \ - \ \hat{D}_{\hat{a}} \hat{\chi} \ - \ \Sigma_{\hat{\mu}}\, \hat{D}_{\hat{a}} Y^{\hat{\mu}}. \tag{2.6}$$

The transformation rules for the non-Abelian scalars X^i get even more corrections: besides transforming as scalars and as coordinates under worldvolume and target space general coordinate transformations respectively, they also transform as adjoint scalars of $U(N)$ and, surprisingly, acquire also a transformation under the NS-NS transformation $\Sigma_{\hat{\mu}}$ [14, 15]:

$$\delta X^{\hat{\mu}} \ = \ \zeta^b \partial_b X^{\hat{\mu}} \ - \ \xi^{\hat{\mu}} \ + \ i[\chi, X^{\hat{\mu}}] \ + \ i\Sigma_{\hat{\rho}}[X^{\hat{\rho}}, X^{\hat{\mu}}]. \tag{2.7}$$

The last term is quite unexpected from the Abelian point of view, as it consists purely of a commutator. Yet it arises very naturally from the T-dualisation of the last term in the variation of δV_x in (2.6), due to the covariant derivative in the pullback,

$$\hat{D}_x Y^i = i[\hat{V}_x, Y^i] \ \longrightarrow \ i[X^{\underline{x}}, X^i], \tag{2.8}$$

as T-duality assumes x to be an isometry direction and hence $\partial_x Y^i = 0$. Note that this mechanism that generates these transformations is exactly the same as the one that generates the dielectric coupling terms in the Chern-Simons term of multiple coinciding D-branes [11]. In a certain sense the NS-NS transformation rules can be though of as "dielectric gauge transformations". It should then be clear that the presence of this term is crucial for the transformation rules for the variations (2.4) to hold also in the non-Abelian case and hence for the consistency of the set-up. It is precisely this term that will change the group structure of the non-Abelian NS-NS transformations.

3 More involved NS-NS transformation rules

In the previous section we saw that form invariance under T-duality of the non-Abelian D-brane actions implies that a gauge transformation of the background Kalb-Ramond field[3] $\delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]}$ in the target space induces a simultaneous transformation of the embeddings scalars and the Born-Infeld vector in the worldvolume, according to

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]}, \qquad \delta X^\mu = \ i\Sigma_\rho[X^\rho, X^\mu], \qquad \delta V_a = \ - \ \Sigma_\mu D_a X^\mu, \tag{3.1}$$

[3]We omit hats on the indices henceforth, as we will not apply T-duality transformations in the rest of the paper. The notation should be self-explicatory

(for technical details on multiplication of algebra elements and non-Abelian functions, we refer to Appendix A).

There are however some derived objects that do not transform as simple as the rules given above. Although they will not yield new insights in the group structure of the NS-NS algebra, that fact that they obey the same algebra will make a strong case for the consistency of our approach.

In general, a non-Abelian function $\Phi(X)$ of the embedding coordinates X^μ transforms under NS-NS gauge transformations as

$$\delta\Phi(X) \;=\; \partial_\mu\Phi(X)\delta X^\mu \;=\; \partial_\mu\Phi(X)\; i\Sigma_\rho[X^\rho, X^\mu] \;=\; i\;\Sigma_\mu[X^\mu, \Phi(X)], \tag{3.2}$$

where in the last step we used the matrix identity (A.6).

The variation of the $U(N)$ covariant derivative D_aX^μ is given by

$$\begin{aligned}
\delta D_aX^\mu \;&=\; \partial_a(\delta X^\mu) \;+\; i[\delta V_a, X^\mu] \;+\; i[V_a, \delta X^\mu] \\
&=\; \partial_a(i\Sigma_\rho[X^\rho, X^\mu]) \;+\; i[-\Sigma_\rho D_aX^\rho, X^\mu] \;+\; i[V_a, i\Sigma_\rho[X^\rho, X^\mu]] \\
&=\; \partial_a(i\Sigma_\rho[X^\rho, X^\mu]) \;+\; i\Sigma_\rho[\partial_aX^\rho, X^\mu] \;+\; i\Sigma_\rho[X^\rho, \partial_aX^\mu] \;+\; i\Sigma_\rho[D_aX^\rho, X^\mu] \\
&\qquad +\; i[\Sigma_\rho, X^\mu]D_aX^\rho \;-\; \Sigma_\rho[V_a, [X^\rho, X^\mu]] \;-\; [V_a, \Sigma_\rho][X^\rho, X^\mu] \\
&=\; i\Sigma_\rho[X^\rho, D_aX^\mu] \;+\; i[X^\mu, \Sigma_\lambda]D_aX^\lambda - i[X^\mu, X^\lambda]D_a\Sigma_\lambda,
\end{aligned} \tag{3.3}$$

where the last two terms can be unified as

$$\delta D_aX^\mu = i\Sigma_\rho[X^\rho, D_aX^\mu] \;+\; 2i[X^\mu, X^\lambda]\partial_{[\lambda}\Sigma_{\nu]}D_aX^\nu. \tag{3.4}$$

The variation of the commutator $[X^\mu, X^\nu]$ can be calculated as

$$\begin{aligned}
\delta[X^\mu, X^\nu] \;&=\; [\delta X^\mu, X^\nu] \;+\; [X^\mu, \delta X^\nu] \\
&=\; [i\Sigma_\rho[X^\rho, X^\mu], X^\nu] \;+\; [X^\mu, i\Sigma_\rho[X^\rho, X^\nu]] \\
&=\; i\Sigma_\rho[[X^\rho, X^\mu], X^\nu] \;+\; i[\Sigma_\rho, X^\nu][X^\rho, X^\mu] \\
&\qquad\qquad +\; [X^\mu, i\Sigma_\rho][X^\rho, X^\nu] \;+\; i\Sigma_\rho[X^\mu, [X^\rho, X^\nu]] \\
&=\; i\Sigma_\rho[X^\rho, [X^\mu, X^\nu]] \;+\; i[X^\mu, \Sigma_\rho][X^\rho, X^\nu] \;-\; i[X^\mu, X^\rho][\Sigma_\rho, X^\nu].
\end{aligned} \tag{3.5}$$

Again the last two terms can be taken together as

$$\delta[X^\mu, X^\nu] \;=\; i\Sigma_\rho[X^\rho, [X^\mu, X^\nu]] \;+\; 2i[X^\mu, X^\rho]\partial_{[\rho}\Sigma_{\lambda]}[X^\lambda, X^\nu]. \tag{3.6}$$

Finally, it will be useful to compute also the variation of a double commutator $[[X^\mu, X^\nu], X^\lambda]$. Using the above result we have that

$$\begin{aligned}
\delta[[X^\mu, X^\nu], X^\lambda] \;&=\; i\Sigma_\rho\Big[X^\rho, [[X^\mu, X^\nu], X^\lambda]\Big] \;-\; i\Big[[X^\mu, X^\nu], X^\rho\Big][\Sigma_\rho, X^\lambda] \\
&+\; i\Big[[X^\mu, X^\nu], \Sigma_\rho\Big][X^\rho, X^\lambda] \;-\; i\Big[[[X^\mu, X^\rho][\Sigma_\rho, X^\nu], X^\lambda\Big] \;+\; i\Big[[[X^\mu, \Sigma_\rho][X^\rho, X^\nu], X^\lambda\Big].
\end{aligned} \tag{3.7}$$

We therefore see that the NS-NS transformations treat embedding scalars, their covariant derivatives and their commutators on different footing, even if they all sit in the adjoint of $U(N)$. A priori there is of course no reason to expect that the NS-NS gauge symmetry would respect the same structures as the $U(N)$ group. In the next section however we will show that there is a certain relation between the two groups.

4 The algebra of NS-NS transformations

A first hint of the group structure of the NS-NS transformations (3.1) comes from realising that there is a class of transformations that leave $B_{\mu\nu}$ invariant, but act non-trivially on the worldvolume fields. Indeed, taking the NS-NS parameter to be exact, $\Sigma_\mu = \partial_\mu \Lambda$, the transformation rules (3.1) take the form

$$\delta B_{\mu\nu} = 0, \qquad \delta X^\mu = \, i\partial_\rho \Lambda [X^\rho, X^\mu], \qquad \delta V_a = \, - \, \partial_\mu \Lambda D_a X^\mu. \tag{4.1}$$

The only non-trivial symmetry in the worldvolume theory is the $U(N)$ gauge group, such that one would expect the above transformations to be $U(N)$ gauge symmetries. Indeed, with the aid of (A.6) and (A.7), these rules can be rewritten as

$$\delta B_{\mu\nu} = 0, \qquad \delta X^\mu = \, i[\Lambda, X^\mu], \qquad \delta V_a = \, - \, D_a \Lambda, \tag{4.2}$$

i.e. as $U(N)$ gauge transformation with parameter Λ. Also the more complicated NS-NS transformation rules for the commutator (3.6) and the covariant derivative (3.4) reduce to the standard transformation rules for objects in the adjoint representation of $U(N)$:

$$\delta [X^\mu, X^\nu] \; = \; i[\Lambda, [X^\mu, X^\nu]], \qquad \delta D_a X^\mu \; = \; i[\Lambda, D_a X^\mu]. \tag{4.3}$$

We therefore see that the pure gauge part of the NS-NS transformations is in fact a $U(N)$ symmetry. However a general NS-NS transformation can not be written as a part of the $U(N)$-algebra and it would be interesting to analyse the complete structure of the intertwining NS-NS and $U(N)$ transformations. We will therefore calculate the commutators of the NS-NS variations amongst each other, NS-NS with $U(N)$ and $U(N)$ with itself.

The last case is in fact trivial, as we are considering the commutation rules of the $U(N)$ (sub-)algebra,

$$[\delta_{\chi_1}, \delta_{\chi_2}] = \delta_{\chi_3} \qquad \text{with} \qquad \chi_3 = i[\chi_1, \chi_2]. \tag{4.4}$$

Less trivial are the commutators involving NS-NS transformations. The mixed $U(N)$ and NS-NS commutator, acting on the scalars, is given by

$$\begin{aligned} [\delta_\Sigma, \delta_\chi] X^\mu \; &= \; \delta_\Sigma \Big(i[\chi, X^\mu] \Big) \; - \; \delta_\chi \Big(i\Sigma_\rho [X^\rho, X^\mu] \Big) \\ &= \; -\Sigma_\rho [X^\rho, [\chi, X^\mu]] + [\chi, X^\rho][\Sigma_\rho, X^\mu] - [\chi, \Sigma_\rho][X^\rho, X^\mu] + [\chi, \Sigma_\rho [X^\rho, X^\mu]]. \end{aligned} \tag{4.5}$$

Using the ($U(N)$) Jacobi identities and the decomposition rules for nested commutators, we see that the above commutator can be written as a $U(N)$ transformation

$$[\delta_\Sigma, \delta_\chi] X^\mu \; = \; i[\tilde{\chi}, X^\mu] \qquad \text{with} \qquad \tilde{\chi} = i\Sigma_\rho [X^\rho, \chi]. \tag{4.6}$$

The same commutator can be checked acting on the Born-Infeld vector V_a to yield

$$[\delta_\Sigma, \delta_\chi] V_a \; = \; - \; D_a \tilde{\chi}, \tag{4.7}$$

with $\tilde{\chi}$ given by the same expression.

Finally, the commutator of two NS-NS transformations is non-trivial as well. Acting on the scalars we find that

$$\begin{aligned} [\delta_{\Sigma^{(1)}}, \delta_{\Sigma^{(2)}}] X^\mu \; &= \; -\Sigma^{(1)}_\lambda [X^\lambda, \Sigma^{(2)}_\rho][X^\rho, X^\mu] \; + \; \Sigma^{(2)}_\lambda [X^\lambda, \Sigma^{(1)}_\rho][X^\rho, X^\mu] \\ &+ i\Sigma^{(2)}_\rho \Big(i\Sigma^{(1)}_\lambda [X^\lambda, [X^\rho, X^\mu]] \; + \; i[X^\rho, \Sigma^{(1)}_\lambda][X^\lambda, X^\mu] \; - \; i[X^\rho, X^\lambda][\Sigma^{(1)}_\lambda, X^\mu] \Big) \\ &+ i\Sigma^{(1)}_\rho \Big(i\Sigma^{(2)}_\lambda [X^\lambda, [X^\rho, X^\mu]] \; + \; i[X^\rho, \Sigma^{(2)}_\lambda][X^\lambda, X^\mu] \; - \; i[X^\rho, X^\lambda][\Sigma^{(2)}_\lambda, X^\mu] \Big) . \end{aligned} \tag{4.8}$$

Given that the first and the second term cancel the seventh and the fifth respectively, this expression can be simplified to

$$\begin{aligned}[\delta_{\Sigma^{(1)}},\delta_{\Sigma^{(2)}}]X^\mu &= \Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\mu,[X^\lambda,X^\rho]] \ + \ [X^\mu,\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho][X^\lambda,X^\rho] \\ &= i\Big[i\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda,X^\rho],X^\mu\Big]. \end{aligned} \tag{4.9}$$

In other words, the commutator of two NS-NS variations is again a $U(N)$ transformation with parameter

$$\bar\chi = i\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda,X^\rho]. \tag{4.10}$$

Again the same result is obtained for the commutator acting on V_a:

$$[\delta_{\Sigma^{(1)}},\delta_{\Sigma^{(2)}}]V_a = \ - \ D_a\left(i\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda,X^\rho]\right). \tag{4.11}$$

We find therefore that the $U(N)$ and the NS-NS transformations form a larger algebra given by

$$\begin{aligned} &[\delta_{\chi_1},\delta_{\chi_2}] = \delta_{\chi_3} && \text{with} \quad \chi_3 = i[\chi_1,\chi_2], \\ &[\delta_\Sigma,\delta_\chi] = \delta_{\tilde\chi} && \text{with} \quad \tilde\chi = i\Sigma_\rho[X^\rho,\chi], \\ &[\delta_{\Sigma_1},\delta_{\Sigma_2}] = \delta_{\bar\chi} && \text{with} \quad \bar\chi = i\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda,X^\rho]. \end{aligned} \tag{4.12}$$

It is worthwhile to check the above algebra on the more complicated transformation rules for the commutator and the covariant derivative. As we mentioned, it will not yield many new insights, but it will give us more confidence in the derived results.

The calculation simplifies greatly if we write in general the NS-NS transformation rules as

$$\delta Z \ = \ \delta^0 Z \ + \ \delta^* Z, \tag{4.13}$$

where $\delta^0 Z$ is the "standard part"

$$\delta^0 Z = i\Sigma_\rho[X^\rho, Z], \tag{4.14}$$

for any Z, and $\delta^* Z$ is the correction terms that appear for Z being $[X^\mu,X^\nu]$ or $D_a X^\mu$,

$$\begin{aligned} \delta^*[X^\mu,X^\nu] &= i[X^\mu,\Sigma_\rho][X^\rho,X^\nu] \ - \ i[X^\mu,X^\rho][\Sigma_\rho,X^\nu], \\ \delta^* D_a X^\mu &= i[X^\mu,\Sigma_\lambda]D_a X^\lambda - i[X^\mu,X^\lambda]D_a\Sigma_\lambda. \end{aligned} \tag{4.15}$$

The trick now is to realise that the δ^* part commutes with both δ^0 and $U(N)$ variations. Indeed, we have that

$$\delta^*(\delta_\chi Z) \ = \ \delta^*\Big(i[\chi,Z]\Big) \ = \ i[\delta^*\chi,Z] \ + \ i[\chi,\delta^* Z] \ = \ \delta_\chi(\delta^* Z), \tag{4.16}$$

as the correction terms δ^* vanish for χ. Hence the mixed commutator $[\delta_\Sigma,\delta_\chi]$ reduces to the standard part

$$[\delta_\Sigma,\delta_\chi]Z \ = \ [\delta^0,\delta_\chi]Z \ + \ [\delta^*,\delta_\chi]Z \ = \ \delta_{\tilde\chi}Z, \tag{4.17}$$

as the second term vanishes identically. Similarly the commutator $[\delta_{\Sigma_1},\delta_{\Sigma_2}]$ simplified to

$$[\delta_{\Sigma_1},\delta_{\Sigma_2}]Z \ = \ [\delta^0,\delta^0]Z \ + \ [\delta^*,\delta^*]Z. \tag{4.18}$$

The first term in the right-hand side is again the standard commutator $\delta_{\bar\chi}Z$, and although the second term has to be calculated for each case separately, it can be shown to vanish identically for both $[X^\mu,X^\nu]$ and $D_a X^\mu$. Hence we see indeed that the algebra (4.12) is also satisfied for these objects, in spite of their involved transformation rules.

5 Jacobi identities

Finally, in order to ensure the consistency of the algebra (4.12), we will check the Jacobi identities. There are four identities to be checked, and from the structure of the algebra it is clear that all will result in a $U(N)$ transformation. We will prove that the Jacobi identities are satisfied by showing that the resulting $U(N)$ transformations have zero parameter.

Indeed, the different identities, acting on a generic object Z, yield $U(N)$ transformations

$$\begin{aligned}
&[\delta_{\chi_1},[\delta_{\chi_2},\delta_{\chi_3}]]Z \ + \ [\delta_{\chi_2},[\delta_{\chi_3},\delta_{\chi_1}]]Z \ + \ [\delta_{\chi_3},[\delta_{\chi_1},\delta_{\chi_2}]]Z \ = \ i[\chi_0, Z],\\
&[\delta_{\Sigma},[\delta_{\chi_1},\delta_{\chi_2}]]Z \ + \ [\delta_{\chi_2},[\delta_{\Sigma},\delta_{\chi_1}]]Z \ + \ [\delta_{\chi_1},[\delta_{\chi_2},\delta_{\Sigma}]]Z \ = \ i[\tilde\chi, Z],\\
&[\delta_{\Sigma_1},[\delta_{\Sigma_2},\delta_{\chi}]]Z \ + \ [\delta_{\chi},[\delta_{\Sigma_1},\delta_{\Sigma_2}]]Z \ + \ [\delta_{\Sigma_2},[\delta_{\chi},\delta_{\Sigma_1}]]Z \ = \ i[\bar\chi, Z],\\
&[\delta_{\Sigma_1},[\delta_{\Sigma_2},\delta_{\Sigma_3}]]Z \ + \ [\delta_{\Sigma_3},[\delta_{\Sigma_1},\delta_{\Sigma_2}]]Z \ + \ [\delta_{\Sigma_2},[\delta_{\Sigma_3},\delta_{\Sigma_1}]]Z \ = \ i[\hat\chi, Z],
\end{aligned} \tag{5.1}$$

with the parameters χ_0, $\tilde\chi$, $\bar\chi$ and $\hat\chi$ given by

$$\begin{aligned}
\chi_0 \ &= \ [\chi_1,[\chi_2,\chi_3]] \ + \ [\chi_2,[\chi_3,\chi_1]] \ + \ [\chi_3,[\chi_1,\chi_2]] \ = \ 0,\\
\tilde\chi \ &= \ \Sigma_\rho[[\chi_1,\chi_2],X^\rho] \ + \ [\chi_1,X^\rho][\Sigma_\rho,\chi_2] \ - \ [\chi_1,\Sigma_\rho][X^\rho,\chi_2]\\
&\qquad - [\chi_2,\Sigma_\rho[X^\rho,\chi_1]] \ + \ [\chi_1,\Sigma_\rho[X^\rho,\chi_2]],\\
\bar\chi \ &= \ -\Sigma^{(1)}_\lambda[X^\lambda,\Sigma^{(2)}_\rho[X^\rho,\chi]] \ + \ \Sigma^{(2)}_\lambda[X^\lambda,\Sigma^{(1)}_\rho[X^\rho,\chi]] \ - \ [\chi,\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda,X^\rho]]\\
&\qquad + \Sigma^{(2)}_\rho[X^\rho,X^\lambda][\Sigma^{(1)}_\lambda,\chi] \ - \ \Sigma^{(2)}_\rho[X^\rho,\Sigma^{(1)}_\lambda][X^\lambda,\chi]\\
&\qquad - \Sigma^{(1)}_\rho[X^\rho,X^\lambda][\Sigma^{(2)}_\lambda,\chi] \ + \ \Sigma^{(1)}_\rho[X^\rho,\Sigma^{(2)}_\lambda][X^\lambda,\chi],\\
\hat\chi \ &= \ -\Sigma^{(1)}_\nu[X^\nu,\Sigma^{(2)}_\lambda\Sigma^{(3)}_\rho[X^\lambda,X^\rho]] \ + \ \Sigma^{(2)}_\lambda\Sigma^{(3)}_\rho[X^\lambda,X^\nu][\Sigma^{(1)}_\nu,X^\rho]\\
&\qquad - \Sigma^{(2)}_\lambda\Sigma^{(3)}_\rho[X^\lambda,\Sigma^{(1)}_\nu][X^\nu,X^\rho]\\
&\quad - \Sigma^{(3)}_\rho[X^\rho,\Sigma^{(1)}_\nu\Sigma^{(2)}_\lambda[X^\nu,X^\lambda]] \ + \ \Sigma^{(1)}_\nu\Sigma^{(2)}_\lambda[X^\nu,X^\rho][\Sigma^{(3)}_\rho,X^\lambda]\\
&\qquad - \Sigma^{(1)}_\nu\Sigma^{(2)}_\lambda[X^\nu,\Sigma^{(3)}_\rho][X^\rho,X^\lambda]\\
&\quad - \Sigma^{(2)}_\lambda[X^\lambda,\Sigma^{(3)}_\rho\Sigma^{(1)}_\nu[X^\rho,X^\nu]] \ + \ \Sigma^{(3)}_\rho\Sigma^{(1)}_\nu[X^\rho,X^\lambda][\Sigma^{(2)}_\lambda,X^\nu]\\
&\qquad - \Sigma^{(3)}_\rho\Sigma^{(1)}_\nu[X^\rho,\Sigma^{(2)}_\lambda][X^\lambda,X^\nu].
\end{aligned} \tag{5.2}$$

The parameter χ_0 vanishes identically due to the Jacobi identity of $U(N)$, but with a little matrix algebra also the other expressions can quite easily be shown to be proportional to the $U(N)$ Jacobi identities and therefore to vanish identically:

$$\begin{aligned}
\tilde\chi \ &= \ \Sigma_\rho\Big([[\chi_1,\chi_2],X^\rho] + [[X^\rho,\chi_1],\chi_2] + [[\chi_2,X^\rho],\chi_]\Big) \ = \ 0,\\
\bar\chi \ &= \ -\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho\Big([X^\lambda,[X^\rho,\chi]] + [\chi,[X^\lambda,X^\rho]] + [X^\rho,[\chi,X^\lambda]]\Big) \ = \ 0,\\
\hat\chi \ &= \ -\Sigma^{(1)}_\nu\Sigma^{(2)}_\lambda\Sigma^{(3)}_\rho\Big([X^\nu,[X^\lambda,X^\rho]] + [X^\rho,[X^\nu,X^\lambda]] + [X^\lambda,[X^\rho,X^\nu]]\Big) \ = \ 0.
\end{aligned} \tag{5.3}$$

6 Conclusions

From the derivation performed in [14, 15] it is known that the background gauge transformation of the Kalb-Ramond field $\delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]}$ induces non-trivial, "dielectric", transformations of the worldvolume field content of the non-Abelian action of a system of multiple coinciding D-branes. These NS-NS transformations acting on the embedding scalars take the form of a pure commutator, $\delta X^\mu = i\Sigma_\rho[X^\rho, X^\mu]$, and arises from T-dualising the transformation rules for the Born-Infeld vector $\delta V_a = -\Sigma_\mu D_a X^\mu$, in the same way as the dielectric couplings arise in the Myers action.

It has been shown already in [15] that the Chern-Simons action of the system of multiple D-branes is invariant under the NS-NS transformations of both the background and the worldvolume fields. In this letter we have shown that the group structure of the NS-NS transformations is more involved than in the Abelian case, due to the fact that the non-Abelian NS-NS transformations rules imply a non-trivial mixture with the $U(N)$ Yang-Mills symmetry.

A first hint of this can be seen if we take the NS-NS parameter to be exact, $\Sigma_\mu = \partial_\mu\Lambda$. In that case, the supergravity part is untouched, $\delta B_{\mu\nu} = 0$, but the worldvolume fields do transform non-trivially under a $U(N)$ gauge transformation with parameter Λ. In other words, the exact part of the NS-NS transformations can be written as a part of the $U(N)$ Yang-Mills symmetry of the worldvolume theory.

However, this reduction to $U(N)$ variations can not be done for a general NS-NS transformation with arbitrary parameter Σ_μ and the full group structure of the intertwining $U(N)$ and NS-NS transformations is revealed by the full algebra

$$\begin{aligned}
&[\delta_{\chi_1}, \delta_{\chi_2}] = \delta_{\chi_3} && \text{with} \quad \chi_3 = i[\chi_1, \chi_2], \\
&[\delta_\Sigma, \delta_\chi] = \delta_{\tilde{\chi}} && \text{with} \quad \tilde{\chi} = i\Sigma_\rho[X^\rho, \chi], \\
&[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{\bar{\chi}} && \text{with} \quad \bar{\chi} = i\Sigma^{(1)}_\lambda\Sigma^{(2)}_\rho[X^\lambda, X^\rho].
\end{aligned} \tag{6.1}$$

We see that the $U(N)$ algebra of the Yang-Mills is a non-trivial sub-algebra of the full algebra, which also involves non-trivial commutators between NS-NS and $U(N)$ transformations and between NS-NS amongst each other. The surprising issue however is that all the resulting commutators turn out to be $U(N)$ transformations, the inherent gauge symmetry of the worldvolume theory. In a certain sense, the $U(N)$ symmetry "non-Abelianises" the other gauge symmetries present (in this case the NS-NS transformations), but only in a very mild way: modulo a $U(N)$ transformation, they still all behave as if they were $U(1)$ gauge symmetries in the Abelian theory.

A few comments are in order: first, as already mentioned in [15], there seems to be a remarkable difference between the NS-NS gauge transformations of $B_{\mu\nu}$ and the R-R transformations of $C_{\mu\nu}$, at least at the level of the worldvolume theory, in spite of the fact that $B_{\mu\nu}$ and $C_{\mu\nu}$ form a doublet under S-duality. The reason for this is of course that S-duality takes us out of the perturbative regime in which we can trust the description we have been working with. However it should be clear that if we believe in S-duality in Type IIB string theory, there should be a "non-Abelianisation" (in the sense of (6.1)) of the R-R gauge transformations, not only of the R-R two-form $C_{\mu\nu}$, but via T-duality of all other R-R potentials as well.

Secondly, it is well known that (parts of the) NS-NS transformations mix with (part of the) general coordinate transformations under T-duality. Knowing on the one hand that a

satisfactory description of general coordinate transformations in non-commutative (or matrix-valued) geometry is still an open issue, and keeping in mind the non-trivial structure of the non-Abelian NS-NS transformations on the other hand, it will be clear that one might find interesting and surprising results on non-commutative geometry and gauge transformations if one tried to T-dualise the algebra (6.1). We leave these ideas for future investigations.

A The non-Abelian formalism

There are many ways to generalise functions $\Phi(x)$ of the Abelian coordinates x^μ to matrix functions $\Phi(X)$ of the matrix-valued coordinates X^μ. One of the most common prescriptions (and the one traditionally used in the context of multiple coinciding D-brane systems) is the symmetrised prescription, through the non-Abelian Taylor expansion of $\Phi(X)$,

$$\Phi(X) = \sum_{n=0}^{\infty} \frac{1}{n!} \, \partial_{\mu_1} \ldots \partial_{\mu_n} \Phi\big|_{x=0} \, X^{\mu_1} \ldots X^{\mu_n}, \tag{A.1}$$

where the matrix multiplication is taken to be the symmetrised product of n Lie algebra elements $A_1, \ldots, A_n$,

$$A_1 \ldots A_n = \frac{1}{n!} \sum_{\sigma \in S_n} A_{\sigma(1)} \ldots A_{\sigma(n)}. \tag{A.2}$$

As the product between X's is not defined within the structure of the algebra, strictly speaking $\Phi(X)$ is not an element of the $u(N)$ algebra. However with the above definitions, $\Phi(X)$ becomes an element of the tensor algebra $T(u(N))$ of $u(N)$. In addition to this, the commutator structure can be transferred to $T(u(N))$, imposing the following equivalence relation

$$AB - BA \sim [A, B], \tag{A.3}$$

yielding the so-called the universal enveloping algebra $U(u(N))$ of $u(N)$). Note that the symmetrised prescription has to be imposed on $U(u(N))$, rather then on $T(u(N))$, due to the equivalence relation (A.3), as otherwise all commutators of the Lie algebra would vanish.

The non-Abelian functions defined in this way have a series of useful properties. For instance, the variation of the non-Abelian functions is given by

$$\delta\Phi(X) \;=\; \sum_{n=0}^{\infty}\sum_{i=1}^{n} \partial_{\mu_1} \ldots \partial_{\mu_n} \Phi\big|_{x=0} \, X^{\mu_1} \ldots \delta X^{\mu_i} \ldots X^{\mu_n} \;=\; \partial_\mu \Phi(X) \delta X^\mu, \tag{A.4}$$

where we used $\partial_\mu \Phi(X)$ as a shorthand for its non-Abelian Taylor expansion. Given that the scalars transform under $u(N)$ gauge transformations as $\delta X^\mu = i[\chi, X^\mu]$, the variation of the non-Abelian function $\Phi(X)$ can then be written as

$$\delta_\chi \Phi(X) \;=\; \partial_\mu \Phi(X) i[\chi, X^\mu] \;=\; i[\chi, \Phi(X)], \tag{A.5}$$

where in the last step we used the properties of the symmetrised prescription and the commutator to prove that

$$[\Phi(X), X^\mu] = \partial_\rho \Phi(X)[X^\rho, X^\mu]. \tag{A.6}$$

In the same way we can also define a covariant derivative of $\Phi(X)$ via

$$D_a \Phi(X) \;=\; \partial_a \Phi(X) + i\,[V_a, \Phi(X)] \;=\; \partial_\mu \Phi(X) D_a X^\mu. \tag{A.7}$$

Similarly the commutator of two non-Abelian functions is given by

$$[\Phi_1(X), \Phi_2(X)] = i[X^\mu, X^\nu]\partial_\mu\Phi_1(X)\partial_\nu\Phi_2(X). \quad \text{(A.8)}$$

It can easily be checked that this definition satisfied the Jacobi identity

$$[\Phi_1, [\Phi_2, \Phi_3]] + [\Phi_2, [\Phi_3, \Phi_1]] + [\Phi_3, [\Phi_1, \Phi_2]] = 0. \quad \text{(A.9)}$$

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